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Quadratic maps and non-prehomogeneous local functional equations

by

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Introduction

Let P and P^* be homogeneous polynomials in n variables of degree d with real coefficients. It is an interesting problem both in Analysis and in Number theory to find a condition on P and P^* under which they satisfy a functional equation, roughly speaking, of the form

$$\text{the Fourier transform of } |P(x)|^s = \text{Gamma factor} \times |P^*(y)|^{-n/d-s} . \tag{1}$$

A beautiful answer to this problem is given by the theory of prehomogeneous vector spaces due to Mikio Sato. Namely, if P and P^* are relative invariants of a regular prehomogeneous vector space and its dual, respectively, and if the characters χ and χ^* corresponding to P and P^*, respectively, satisfy the relation $\chi\chi^* = 1$, then, P and P^* satisfy a functional equation (see [18], [19], [12]). The theory works quite satisfactorily and it might give an impression that prehomogeneous vector spaces are the final answer to the problem.

Meanwhile, in [6], Faraut and Koranyi developed a method of constructing polynomials with the property (1), starting from representations of Euclidean (formally real) Jordan algebras. What is remarkable in their result is that, from representations of simple Jordan algebras of rank 2, one can obtain a series of polynomials satisfying (1), which are not covered by the theory of prehomogeneous vector spaces. Their result was later generalized by Clerc [5]. Thus we got to know that the class of polynomials with the property (1) is broader than the class of relative invariants of regular prehomogeneous vector spaces.

Now the ultimate goal of our investigation should be the characterization of polynomials P and P^* with the property (1). The purpose of the present paper is, however, much more modest; we give a new construction of polynomials with the property (1), which includes the result of Faraut, Koranyi and Clerc as a special case.

Our main result may be outlined as follows: Suppose that we are given homogeneous polynomials P and P^* on a real vector spaces V and its dual V^*, respectively, satisfying a functional equation of the form (1). Further suppose that there exists a nondegenerate quadratic mapping Q (resp. Q^*) of another real vector space W (resp. W^*) to V (resp. V^*), and Q and Q^* are dual. Then, the polynomials $\tilde{P} = P \circ Q$ and $\tilde{P}^* = P^* \circ Q^*$ inherit the property (1) from P and P^* and the gamma factors for the new functional equation have an explicit expression in term of those for P and P^*. We refer to Section 1 for a precise formulation of the main result (Theorem 4), which includes the case of local zeta functions in several variables.

The prototype of our result is the case where $V = V^* = \mathbb{R}$, $P(x) = x$, $P^*(y) = y$, $W = W^* = \mathbb{R}^m$, and $\tilde{P} = Q$ and $\tilde{P}^* = Q^*$ are nondegenerate quadratic forms dual to each other. This case was dealt with by Rallis and Schiffmann [11] and they derived an explicit functional equation of the form (1) from the functional equation satisfied by $|x|^s$ (see Section 2.1). Our main result is proved in Section 3 by combining the method of Rallis and Schiffmann with a trick used by Shintani ([19]) in the proof of the local functional equations in the theory of prehomogeneous vector spaces.

Section 2 is devoted to a discussion of examples. In Section 2.2, we explain the result of Faraut-Koranyi from our point of view and, in Section 2.3, we give some standard examples of quadratic mappings from prehomogenous vector spaces to another prehomogeneous vector spaces for which our main theorem can apply. If we take $V = V^* = \mathbb{R}^n$ and nondegenerate quadratic forms, which are dual to each other, on V and V^* as P and P^*, then one can construct non-degenerate dual quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$ and we can obtain several new examples of polynomials satisfying functional equations that do not come from prehomogenous vector spaces. In this case the explicit form of the functional equation for $\tilde{P} = P \circ Q$ and $\tilde{P}^* = P^* \circ Q^*$ can be obtained from the 1-dimensional case by repeated applications of the main result. The non-prehomogeneous polynomials with the property (1) appearing in the work of Faraut, Koranyi and Clerc is a special case where the signature of the quadratic forms P and P^* is $(1, n-1)$. In Section 2.4, for any even positive integer $n \geq 4$, we give such quadratic mappings in the case where the signature of the quadratic forms is $(n/2, n/2)$. For $n = 4, 6$, we obtain quadratic mappings between two prehomogeneous vector spaces; however, if $n \geq 8$, we obtain new examples of non-prehomogeneous functional equations. The general case of arbitrary signature will be dealt with in a joint work with T. Kogiso.

It is natural to ask whether global zeta functions with functional equations can be associated with polynomials $\tilde{P}$ and $\tilde{P}^*$ given in our main theorem. For polynomials obtained from the theory of Faraut and Koranyi, this problem was solved by Achab in [1] and [2]. The problem is open in our general setting.

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1. Statement of Main result

1.1. Local functional equations

Let V be a real vector space of dimension n and V^* the vector space dual to V. Let $P_1, \dots, P_r$ (resp. $P_1^*, \dots, P_r^*$) be $\mathbb{R}$-irreducible homogeneous polynomials on V (resp. V^*). We put

$$\Omega = \{v \in V \mid P_1(v) \cdots P_r(v) \neq 0\} \quad \text{and}$$
$$\Omega^* = \{v* \in V^* \mid P_1^*(v^*) \cdots P_r^*(v^*) \neq 0\}.$$

We assume that

(A.1) there exists a biregular rational mapping $\phi : \Omega \to \Omega^*$ defined over $\mathbb{R}$.

Let

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_\nu, \quad \Omega^* = \Omega_1^* \cup \cdots \cup \Omega_\nu^*$$

be the decomposition into connected components of Ω and Ω^*. Note that (A.1) implies that the numbers of connected components of Ω and Ω^* are same and we may assume that

$$\Omega_j^* = \phi(\Omega_j) \quad (j = 1, \dots, \nu).$$

For an $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ with $\Re(s_1), \dots, \Re(s_r) > 0$, we define a continuous function $|P(v)|_j^s$ on V by

$$|P(v)|_j^s = \begin{cases} \prod_{i=1}^r |P_i(v)|^{s_i}, & v \in \Omega_j, \\ 0, & v \notin \Omega_j. \end{cases}$$

The function $|P(v)|_j^s$ can be extended to a tempered distribution depending on s in $\mathbb{C}^r$ meromorphically. For an $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, we put

$$P^{\mathbf{m}}(v) = \prod_{i=1}^r P_i(v)^{m_i}.$$

Sometimes we use the symbol $P^{\mathbf{m}}(v)$ for non-integral $\mathbf{m}$ (see the second identity in Lemma 1), which we may regard either as a symbolic expression on which differential operators operate in a usual manner or as a function on the universal covering space of Ω. Similarly we define $|P^*(v^*)|_j^s$ ($s \in \mathbb{C}^r$) and $P^{*\mathbf{m}}(v^*)$ ($\mathbf{m} \in \mathbb{Z}^r$). The homogeneous degree of $P^{\mathbf{m}}$ (resp. $P^{*\mathbf{m}}$) is denoted by $d(\mathbf{m})$ (resp. $d^*(\mathbf{m})$).

We define $\epsilon_i(\mathbf{m})$ (resp. $\epsilon_j^*(\mathbf{m})$) to be the sign of values of $P^{\mathbf{m}}$ (resp. $P^{*\mathbf{m}}$) on Ω_i (resp. Ω_j^*). Namely we put

$$\epsilon_i(\mathbf{m}) = P^{\mathbf{m}}(v)/|P^{\mathbf{m}}(v)| \quad (v \in \Omega_i), \quad \epsilon_j^*(\mathbf{m}) = P^{*\mathbf{m}}(v^*)/|P^{*\mathbf{m}}(v^*)| \quad (v^* \in \Omega_j^*).$$

Since Ω_i and Ω_j^* are assumed to be connected, $\epsilon_i(\mathbf{m})$ and $\epsilon_j^*(\mathbf{m})$ do not depend on the choice of v and v^*.

We denote by $\mathcal{S}(V)$ and $\mathcal{S}(V^*)$ the spaces of rapidly decreasing functions on the real vector spaces V and V^*, respectively. For $\Phi \in \mathcal{S}(V)$ and $\Phi^* \in \mathcal{S}(V^*)$, we define the local zeta functions by setting

$$\zeta_i(s, \Phi) = \int_V |P(v)|_i^s \Phi(v) dv\,, \zeta_i^*(s, \Phi^*) = \int_{V^*} |P^*(v^*)|_i^s \Phi^*(v^*)\, dv^* \quad (i = 1, \ldots, \nu)\,,$$

where dv and dv^* are the Euclidean measures dual to each other. It is well-known that the local zeta functions $\zeta_i(s, \Phi)$, $\zeta_i^*(s, \Phi^*)$ are absolutely convergent for $\Re(s_1), \ldots, \Re(s_r) > 0$ and have analytic continuations to meromorphic functions of s in $\mathbb{C}^r$. We assume the following:

(A.2) There exist an $A \in GL_r(\mathbb{Z})$ and a $\lambda \in \mathbb{C}^r$ such that a functional equation of the form

$$\zeta_i^*((s+\lambda)A, \hat{\Phi}) = \sum_{j=1}^{\nu} \Gamma_{ij}(s)\zeta_j(s, \Phi) \quad (i = 1, \ldots, \nu) \tag{2}$$

holds for every $\Phi \in \mathcal{S}(V)$, where $\Gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}^r$ not depending on Φ with $\det(\Gamma_{ij}(s)) \not\equiv 0$ and $\hat{\Phi}$ is the Fourier transform of Φ:

$$\hat{\Phi}(v^*) = \int_V \Phi(v) \exp(-2\pi\sqrt{-1}\langle v, v^*\rangle)\, dv\,.$$

A lot of examples of $\{P_1, \ldots, P_r\}$ and $\{P_1^*, \ldots, P_r^*\}$ satisfying (A.1) and (A.2) can be obtained from relative invariants of regular prehomogeneous vector spaces (see [15], [18], [12]). However, we do not assume here the existence of group action that relates the polynomials to prehomogeneous vector spaces.

LEMMA 1. *Assume that the assumption* (A.2) *is satisfied. For* $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ *with* $m_1, \ldots, m_r \geq 0$, *denote by* $P^{*\mathbf{m}}(\partial_v)$ *the linear partial differential operator with constant coefficients satisfying*

$$P^{*\mathbf{m}}(\partial_v) \exp(\langle v, v^*\rangle) = P^{*\mathbf{m}}(v^*) \exp(\langle v, v^*\rangle)\,.$$

Then, there exists a polynomial $b_{\mathbf{m}}(s)$ *of* $s_1, \ldots, s_r$ *such that*

$$P^{*\mathbf{m}}(\partial_v) P^s(v) = b_{\mathbf{m}}(s) P^{s+\mathbf{m}'}(v), \quad \mathbf{m}' = \mathbf{m}A^{-1}\,.$$

Moreover, if $\Gamma_{ij}(s)$ *does not vanish identically, then the polynomial* $b_{\mathbf{m}}(s)$ *is given by*

$$b_{\mathbf{m}}(s) = (-2\pi\sqrt{-1})^{d^*(\mathbf{m})} \epsilon_j(\mathbf{m}')\epsilon_i^*(\mathbf{m}) \cdot \frac{\Gamma_{ij}(s+\mathbf{m}')}{\Gamma_{ij}(s)}\,.$$

We call $b_{\mathbf{m}}(s)$ the b-functions of $\{P_1, \ldots, P_r\}$. By the last identity in the lemma, we can define $b_{\mathbf{m}}(s)$ for any $\mathbf{m} \in \mathbb{Z}^r$. The b-functions satisfy the cocycle property

$$b_{\mathbf{m}+\mathbf{n}}(s) = b_{\mathbf{m}}(s) b_{\mathbf{n}}(s+\mathbf{m}') \quad (\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r)\,. \tag{3}$$

The lemma says that the existence of b-functions is a necessary condition for local functional equations. We postpone the proof of this lemma to Section 3.1 (see Lemma 9 and its proof).

1.2. Nondegenerate dual quadratic mappings

Let W be a real vector space with dimension m and W^* the vector space dual to W. Suppose that we are given quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$. The mappings $B_Q : W \times W \to V$ and $B_{Q^*} : W^* \times W^* \to V^*$ defined by

$$B_Q(w_1, w_2) := Q(w_1 + w_2) - Q(w_1) - Q(w_2)\,,$$
$$B_{Q^*}(w_1^*, w_2^*) := Q^*(w_1^* + w_2^*) - Q^*(w_1^*) - Q^*(w_2^*)$$

are bilinear. For given $v \in V$ and $v^* \in V^*$, the mappings $Q_{v^*} : W \to \mathbb{R}$ and $Q_v^* : W^* \to \mathbb{R}$ defined by

$$Q_{v^*}(w) = \langle Q(w), v^* \rangle, \quad Q_v^*(w^*) = \langle v, Q^*(w^*) \rangle$$

are quadratic forms on W and W^*, respectively. We assume that Q and Q^* are nondegenerate and dual to each other with respect to the biregular mapping ϕ in (A.1). This means that Q and Q^* satisfy the following:

(A.3) (i) (Nondegeneracy) The algebraic set $\tilde{\Omega} := Q^{-1}(\Omega)$ (resp. $\tilde{\Omega}^* = Q^{*-1}(\Omega^*)$) is open dense in W (resp. W^*) and the rank of the differential of Q (resp. Q^*) at $w \in \tilde{\Omega}$ (resp. $w^* \in \tilde{\Omega}^*$) is equal to n. (In particular, $m \geq n$.)
(ii) (Duality) For any $v \in \Omega$, the quadratic forms $Q_{\phi(v)}$ and Q_v^* are dual to each other. Namely, fix a basis of W and the basis of W^* dual to it, and denote by S_{v^*} and S_v^* the matrices of the quadratic forms Q_{v^*} and Q_v^* with respect to the bases. Then $S_{\phi(v)}$ and S_v^* $(v \in \Omega)$ are nondegenerate and $S_{\phi(v)} = (S_v^*)^{-1}$.

Now we collect some elementary consequences of the assumptions (A.1) and (A.3). First note that a rational function defined over $\mathbb{R}$ with no zeros and no poles on Ω (resp. Ω^*) is a monomial of $P_1, \ldots, P_r$ (resp. $P_1^*, \ldots, P_r^*$). Hence the assumptions (A.1) and (A.3) (ii) imply the following lemma.

LEMMA 2. *If we replace P_i, P_j^*, ϕ by their suitable real constant multiples (if necessary),*

(1) *there exists a $B = (b_{ij}) \in GL_r(\mathbb{Z})$ such that*

$$P_i^*(\phi(v)) = \prod_{j=1}^{r} P_j(v)^{b_{ij}} \quad (i = 1, \ldots, r)\,.$$

(2) *There exist $\kappa, \kappa^* \in \mathbb{Z}^r$ and a non-zero constant α such that*

$$\det S_v^* = \alpha^{-1} P^{\kappa}(v)\,, \quad \det S_{v^*} = \alpha P^{*\kappa^*}(v^*)\,.$$

(3) *There exists a $\mu \in \mathbb{Z}^r$ such that*

$$\det\left(\frac{\partial \phi(v)_i}{\partial v_j}\right) = \pm P^{\mu}(v)\,.$$

If $P_1, \ldots, P_r$ and $P_1^*, \ldots, P_r^*$ are the fundamental relative invariants of a regular prehomogeneous vector space (G, ρ, V) and its dual (G, ρ^*, V^*), then we have $B = A^{-1}$. Indeed, by the regularity, there exists a relative invariant P for which $\phi(v) = \operatorname{grad} \log P$ is a G-equivariant morphism satisfying (A.1). From the G-equivariance of the mapping ϕ

([17, §4, Prop. 9]), we have $B = A^{-1}$ (see [12]). It is very likely that the identity $B = A^{-1}$ always holds under the assumption (A.1) and (A.2) and, for simplicity, we assume

(A.4) $B = A^{-1}$.

LEMMA 3. *The mapping* $\phi : \Omega \to \Omega^*$ *is homogeneous of degree* -1. *Namely we have* $\phi(tv) = t^{-1}\phi(v)$ *for* $t \in \mathbb{R} \setminus \{0\}$ *and* $v \in \Omega$.

Proof. Fix bases of V and W and identify them with $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then there exist real symmetric matrices $S_1, \dots, S_n$ such that $Q(w) = ({}^t w S_1 w, \dots, {}^t w S_n w)$. If $S_1, \dots, S_n$ are linearly dependent, then the image $Q(W)$ is contained in a hyperplane in V. This contradicts the nondegeneracy of Q. Hence $S_1, \dots, S_n$ are linearly independent and the identity $S_{v_1^*} = S_{v_2^*}$ implies that $v_1^* = v_2^*$. By the duality, we have

$$S_{\phi(tv)} = (S_{tv}^*)^{-1} = (tS_v^*)^{-1} = t^{-1}(S_v^*)^{-1} = t^{-1}S_{\phi(v)} = S_{t^{-1}\phi(v)} \quad (v \in \Omega, t \in \mathbb{R}) .$$

By the observation above, we obtain $\phi(tv) = t^{-1}\phi(v)$. □

Since we assumed that Ω_i (resp. Ω_i^*) are connected components, the signature of the quadratic form $Q_v^*(w^*)$ (resp. $Q_{v^*}(w)$) on W^* (resp. W) do not change when v (resp. v^*) varies on Ω_i (resp. Ω_i^*). Let p_i and q_i be the numbers of positive and negative eigenvalues of Q_v^* for $v \in \Omega_i$ and put

$$\gamma_i = \exp\left(\frac{(p_i - q_i)\pi\sqrt{-1}}{4}\right) \quad (i = 1, \dots, \nu) , \tag{4}$$

which are 8th roots of unity. For $\Psi \in \mathcal{S}(W)$, we denote by $\check{\Psi}$ the (inverse) Fourier transform of Ψ:

$$\check{\Psi}(w^*) = \int_W \Psi(w) \exp(2\pi\sqrt{-1}\langle w, w^*\rangle)\, dw .$$

Then, by (A.3) (ii) and the celebrated identity by Weil ([20, n°14, Théorème 2]), we have

$$\begin{aligned}
&\int_{W^*} \exp(2\pi\sqrt{-1}Q_v^*(w^*))\check{\Psi}(w^*)\, dw^* \\
&\quad = 2^{-m/2}|\alpha|^{1/2}\gamma_i |P(v)|_i^{-\kappa/2} \\
&\qquad \times \int_W \exp\left(-\frac{\pi\sqrt{-1}}{2} \cdot Q_{\phi(v)}(w)\right)\Psi(w)\, dw \quad (v \in \Omega_i, \Psi \in \mathcal{S}(W)) ,
\end{aligned} \tag{5}$$

where dw and dw^* are the Euclidean measures dual to each other. This identity is the key to the proof of our main theorem.

REMARK. For simplicity we assumed that Ω_i are connected components. However, in the following discussion, it is sufficient to assume that $\epsilon_i(m)$ and γ_i are well defined for Ω_i.

1.3. Main theorem

We put

$$\tilde{P}_i(w) = P_i(Q(w)), \quad \tilde{P}_i^*(w^*) = P_i^*(Q^*(w^*)) \quad (i = 1, \dots, r)$$
$$\tilde{\Omega}_i = Q^{-1}(\Omega_i), \quad \tilde{\Omega}_i^* = Q^{*-1}(\Omega_i^*) \quad (i = 1, \dots, \nu).$$

Some of $\tilde{\Omega}_i$'s and $\tilde{\Omega}_i^*$'s may be empty. We define $|\tilde{P}(w)|_i^s$ and $|\tilde{P}^*(w^*)|_i^s$ in the same manner as in Section 1.1. The zeta functions associated with these polynomials are defined by

$$\tilde{\zeta}_i(s, \Psi) = \int_W |\tilde{P}(w)|_i^s \Psi(w)\, dw, \quad \tilde{\zeta}_i^*(s, \Psi^*) = \int_{W^*} |\tilde{P}^*(w^*)|_i^s \Psi^*(w^*)\, dw^*.$$

Then our main result is that the functional equation (2) for P_i's and P_j^*'s implies a functional equation for $\tilde{P}_i$'s and $\tilde{P}_j^*$'s and the gamma factors in the new functional equation can be written explicitly. Namely, we have the following theorem.

THEOREM 4. *Under the assumptions* (A.1)–(A.4), *the zeta functions* $\tilde{\zeta}_i(s, \Psi)$ *and* $\tilde{\zeta}_i^*(s, \Psi^*)$ *satisfy the functional equation*

$$\tilde{\zeta}_i^*((s + 2\lambda + \kappa/2 + \mu)A, \check{\Psi}) = \sum_{j=1}^{\nu} \tilde{\Gamma}_{ij}(s)\tilde{\zeta}_j(s, \Psi),$$

where the gamma factors $\tilde{\Gamma}_{ij}(s)$ *are given by*

$$\tilde{\Gamma}_{ij}(s) = 2^{-2d(s)-m/2}|\alpha|^{1/2} \sum_{k=1}^{\nu} \gamma_k \Gamma_{ik}(s + \lambda + \kappa/2 + \mu)\Gamma_{kj}(s).$$

Here we denote by $d(s)$ $(s \in \mathbb{C}^r)$ *the homogeneous degree of* P^s*, namely,* $d(s) = \sum_{i=1}^r s_i \deg P_i$.

By Lemma 1, we have the following formula expressing the b-functions $\tilde{b}_{\mathbf{m}}(s)$ of $\{\tilde{P}_1, \dots, \tilde{P}_r\}$ in terms of the b-functions $b_{\mathbf{m}}(s)$ of $\{P_1, \dots, P_r\}$.

COROLLARY TO THEOREM 4. *For* $\mathbf{m} \in \mathbb{Z}^r$, *we have*

$$\tilde{b}_{\mathbf{m}}(s) = b_{\mathbf{m}}(s) b_{\mathbf{m}}(s + \lambda + \kappa/2 + \mu)$$

up to a constant multiple.

In the case of one variable zeta functions, namely, in the case of $r = 1$, writing $P = P_1$ and $P^* = P_1^*$, we have the following lemma.

LEMMA 5. *Assume that* $r = 1$. *Then we have*

$$A = B = -1, \quad d := \deg P = \deg P^*, \quad \lambda = \frac{n}{d}, \quad \mu = -\frac{2n}{d}, \quad \kappa = \frac{m}{d}.$$

Proof. For $t > 0$, put $\Phi_t(v) := \Phi(tv)$ $(\Phi \in \mathcal{S}(V))$. Then we have

$$\zeta_j(s, \Phi_t) = t^{-ds-n}\zeta_j(s, \Phi) \quad \text{and} \quad \zeta_i^*((s + \lambda)A, \widehat{\Phi_t}) = t^{d^*(s+\lambda)A}\zeta_i^*((s + \lambda)A, \hat{\Phi}),$$

where $d = \deg P$ and $d^* = \deg P^*$. Hence the functional equation (2) implies that $-d = d^*A$ and $-n = d^*\lambda A$. Since $A \in GL_1(\mathbb{Z}) = \{\pm 1\}$, we have $A = -1$, $d = d^*$, $\lambda = n/d$. Moreover, by Lemma 3, we have $-d^* = dB$ and $-2n = d\mu$, which imply that $B = -1$ and $\mu = -2n/d$. By Lemma 2 (2), $P^{\kappa}(v) = \det S_v^*$ is the determinant of a matrix depending on v linearly and is homogeneous of degree m. Hence $\kappa = m/d$. □

By Lemma 5, if $r = 1$, then the functional equation for local zeta functions takes the form

$$\tilde{\zeta}_i^*\left(-s-\frac{m}{2d}, \check{\Psi}\right) = \sum_{j=1}^{\nu} \tilde{\Gamma}_{ij}(s)\tilde{\zeta}_j(s, \Psi),$$

$$\tilde{\Gamma}_{ij}(s) = 2^{-2ds-m/2}|\alpha|^{1/2} \sum_{k=1}^{\nu} \gamma_k \Gamma_{ik}\left(s + \frac{m-2n}{2d}\right)\Gamma_{kj}(s) \tag{6}$$

and the b-function is given by

$$\tilde{b}(s) = b(s)b\left(s + \frac{m-2n}{2d}\right), \tag{7}$$

where $b(s)$ and $\tilde{b}(s)$ are defined by

$$P^*(\partial_v)P^{s+1}(v) = b(s)P^s(v), \quad \tilde{P}^*(\partial_w)\tilde{P}^{s+1}(w) = \tilde{b}(s)\tilde{P}^s(v).$$

(For general $\mathbf{m} \in \mathbb{Z}$, the b-function $b_{\mathbf{m}}(s)$ (resp. $\tilde{b}_{\mathbf{m}}(s)$) can be determined from $b(s)$ (resp. $\tilde{b}(s)$) by using the cocycle property (3).)

Before proving Theorem 4, we give in the next section several examples of quadratic mappings to which the theorem applies.

2. Examples

As we remarked in Section 1.1, the theory of prehomogeneous vector spaces gives examples of polynomials satisfying the assumptions (A.1) and (A.2). If we are given non-degenerate dual quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$ over a regular prehomogeneous vector space (G, ρ, V) and its dual (G, ρ^*, V^*), then by Theorem 4, we obtain new polynomials on W and W^* satisfying a similar local functional equation. In Theorem 4, we did not assume that $P_1, \ldots, P_r$ and $P_1^*, \ldots, P_r^*$ admit a prehomogeneous group action. However, all the examples we have on hand, some of which will be discussed below, are quadratic mappings to prehomogeneous vector spaces. For our later use, we give here a criterion for the nondegeneracy and the duality for equivariant quadratic mappings to prehomogeneous vector spaces.

Let (G, ρ, V) be a regular prehomogeneous vector space defined over $\mathbb{R}$ and (G, ρ^*, V^*) the dual prehomogeneous vector space. We assume that the singular set S (resp. S^*) of (G, ρ, V) (resp. (G, ρ^*, V^*)) is a hypersurface. We decompose S (resp. S^*) into the union of $\mathbb{R}$-irreducible hypersurfaces and let $P_1, \ldots, P_r$ (resp. $P_1^*, \ldots, P_r^*$) be the $\mathbb{R}$-irreducible polynomials defining the irreducible components. The polynomials $P_1, \ldots, P_r$ (resp. $P_1^*, \ldots, P_r^*$) are unique up to constant multiples and are called the *fundamental* relative invariants over $\mathbb{R}$ of (G, ρ, V) (resp. (G, ρ^*, V^*)). The regularity of the prehomogeneous vector spaces implies that there exists a relative invariant P_0 defined over $\mathbb{R}$ such that the rational mapping $\phi : V \to V^*$ defined by $\phi(v) = \operatorname{grad} \log P_0(v)$ induces a biregular morphism of $\Omega = V - S$ onto $\Omega^* = V^* - S^*$. The mapping ϕ is G-equivariant: $\phi(\rho(g)v) = \rho^*(g)\phi(v)$ $(v \in \Omega,\ g \in G)$.

Let $\tau : G \to GL(W)$ be a rational representation of G defined over $\mathbb{R}$ on a vector space W with $\mathbb{R}$-structure and $\tau^* : G \to GL(W^*)$ the contragredient representation on the vector space W^* dual to W. Let $Q : W \to V$ and $Q^* : W^* \to V^*$ be G-equivariant quadratic mappings:

$$Q(\tau(g)w) = \rho(g)Q(w)\,, \quad Q^*(\tau^*(g)w^*) = \rho^*(g)Q^*(w^*)\,.$$

In the following we identify G, V, V^*, W, W^*, Ω, Ω^* with their sets of real points.

LEMMA 6. (1) *If none of $\tilde{P}_i(w) = P_i(Q(w))$ (resp. $\tilde{P}_i^*(w^*) = P_i^*(Q^*(w^*))$) $(i = 1, \dots, r)$ vanish identically, then the mapping Q (resp. Q^*) is nondegenerate.*

(2) *If $Q_v^*(w^*)$ and $Q_{\phi(v)}(w)$ are dual to each other for a $v \in \Omega$, then they are dual to each other for all $v \in \Omega$.*

Proof. (1) From the G-equivariance, the image of the differential dQ_w of Q at $w \in W$ contains $d\rho(\mathrm{Lie}(G))Q(w)$. By the non-vanishing of $\tilde{P}_i$, $\tilde{\Omega} = Q^{-1}(\Omega)$ is an open dense subset of W. If $w \in \tilde{\Omega}$, then $Q(w)$ is in the open G-orbit Ω and $d\rho(\mathrm{Lie}(G))Q(w) = V$. This proves the nondegeneracy.

(2) Assume that $Q_{v_0}^*(w^*)$ and $Q_{\phi(v_0)}(w)$ are dual to each other for a $v_0 \in \Omega$. We have to prove the identity $(S_v^*)^{-1} = S_{\phi(v)}$ for every $v \in \Omega$. Since this matrix identity is an algebraic identity, it is sufficient to prove it for an arbitrary point v in the open orbit $\rho(G)v_0$. Take a $g \in G$ such that $v = \rho(g)v_0$. Then

$$\begin{aligned} Q_v^*(w^*) &= \langle \rho(g)v_0\,,\, Q^*(w^*)\rangle = \langle v_0\,,\, \rho^*(g^{-1})Q^*(w^*)\rangle \\ &= \langle v_0\,,\, Q^*(\tau^*(g^{-1})w^*)\rangle = Q_{v_0}^*(\tau^*(g^{-1})w^*)\,. \end{aligned}$$

Similarly we have

$$Q_{\phi(v)}(w) = Q_{\phi(\rho(g)v_0)}(w) = Q_{\rho(g)\phi(v_0)}(w) = Q_{\phi(v_0)}(\tau(g^{-1})w)\,.$$

We identify V and V^* (resp. W and W^*) with $\mathbb{R}^n$ (resp. $\mathbb{R}^m$) via dual bases. Then the matrices $S_{v_0}^*$ and $S_{\phi(v_0)}$ of $Q_{v_0}^*(w^*)$ and $Q_{\phi(v_0)}(w)$ satisfy the relation $(S_{v_0}^*)^{-1} = S_{\phi(v_0)}$. By the identities above, we have $S_v^* = {}^t\tau^*(g^{-1})S_{v_0}^*\tau^*(g^{-1})$ and $S_{\phi(v)} = {}^t\tau(g^{-1})S_{\phi(v_0)}\tau(g^{-1})$. By our identification of W and W^* with $\mathbb{R}^m$, we have $\tau^*(g) = {}^t\tau(g)^{-1}$. Hence we obtain $(S_v^*)^{-1} = S_{\phi(v)}$. □

2.1. Quadratic forms

The simplest example is the case where

$$\begin{aligned} V = V^* = \mathbb{R}\,, \quad P(v) = P^*(v) = v\,, \quad \phi : v \mapsto v^{-1}\,, \\ W = W^* = \mathbb{R}^m\,, \quad Q(w) = {}^t wYw\,, \quad Q^*(w) = {}^t wY^{-1}w\,. \end{aligned}$$

Here Y is a nondegenerate real symmetric matrix of size m. Then,

$$P^*(\phi(v)) = P(v)^{-1}\,, \quad \det S_{v^*} = |\det Y|^{-1} P^*(v^*)\,,$$
$$S_v^* = |\det Y| P(v)\,, \quad \frac{d\phi}{dv} = -P(v)^{-2}\,.$$

We may put

$$\Omega_1 = \Omega_1^* = \{v \in \mathbb{R} \mid v > 0\}\,, \quad \Omega_2 = \Omega_2^* = \{v \in \mathbb{R} \mid v < 0\}\,.$$

Then, $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega^* = \Omega_1^* \cup \Omega_2^*$. The functional equation satisfied by P and P^* is

$$\begin{pmatrix} \zeta_1(s, \hat{\Phi}) \\ \zeta_2(s, \hat{\Phi}) \end{pmatrix} = (2\pi)^{-s-1}\Gamma(s+1) \begin{pmatrix} e^{-\pi\sqrt{-1}(s+1)/2} & e^{\pi\sqrt{-1}(s+1)/2} \\ e^{\pi\sqrt{-1}(s+1)/2} & e^{-\pi\sqrt{-1}(s+1)/2} \end{pmatrix} \begin{pmatrix} \zeta_1(-1-s, \Phi) \\ \zeta_2(-1-s, \Phi) \end{pmatrix}\,.$$

If Y has p positive and $m-p$ negative eigenvalues, then $\gamma_1 = e^{\pi\sqrt{-1}(2p-m)/4}$ and $\gamma_2 = e^{\pi\sqrt{-1}(m-2p)/4}$. Hence, by the identity (6) (the $r = 1$ case of Theorem 4), we have the following functional equation for $\tilde{P}(w) = Q(w)$ and $\tilde{P}^*(w) = Q^*(w)$:

$$\begin{aligned} \begin{pmatrix} \tilde{\zeta}_1(s, \check{\Psi}) \\ \tilde{\zeta}_2(s, \check{\Psi}) \end{pmatrix} &= \pi^{-2s-\frac{m}{2}-1} |\det Y|^{1/2} \Gamma(s+1)\Gamma\left(s + \frac{m}{2}\right) \\ &\quad \times \begin{pmatrix} -\sin\frac{\pi}{2}(m-p+2s) & \sin\frac{p\pi}{2} \\ \sin\frac{(m-p)\pi}{2} & -\sin\frac{\pi}{2}(p+2s) \end{pmatrix} \begin{pmatrix} \tilde{\zeta}_1(-\frac{m}{2}-s, \Psi) \\ \tilde{\zeta}_2(-\frac{m}{2}-s, \Psi) \end{pmatrix}. \end{aligned} \tag{8}$$

The proof of this well-known formula can be found in [7] (see also [9, Prop. 4.27]) and [11]. The method of the proof given in Section 3 of our main theorem is a generalization of that of Rallis and Schiffmann [11].

As will be explained in Sections 2.2 and 2.4, one can construct nondegenerate quadratic mappings into quadratic spaces and obtain homogeneous polynomials of degree 4 satisfying local functional equations. The gamma matrices of the functional equations satisfied by such polynomials of degree 4 are given explicitly by Theorem 4 and the identity (8).

2.2. Representations of Euclidean Jordan Algebras

In [6, Chap. 8], Faraut and Koranyi proved that, starting from a representation of a Euclidean Jordan algebra, one can construct polynomials satisfying local functional equations. Their result was later generalized by Clerc [5] to zeta functions of several variables. Here we explain how their results can be incorporated in our Theorem 4.

Let V be a real simple Euclidean Jordan algebra with unity e, of dimension n and rank r. Denote by $P(v) = \det v$ the generic norm of V. Then $\Omega := \{v \in V \mid \det v \neq 0\}$ coincides with the set $V^\times$ of invertible elements in V. Let Ω_1 be the connected component of Ω containing e, the symmetric cone associated with V. Let G be the identity component of the group of linear transformations that preserve Ω_1, which is a real reductive Lie group. Then it is known that (G, V) is (a real form of) a prehomogeneous vector space, and the norm $P(v) = \det v$ of V is its fundamental relative invariant. More generally, restricting the G-action on V to the action of a minimal parabolic subgroup of G, we still have a prehomogeneous vector space with r fundamental relative invariants of minor determinant type. The prehomogeneous vector space is regular and we obtain a local functional equation (A.2) for zeta functions of r variables. Moreover the mapping $\phi : \Omega \to \Omega$ defined by $\phi(v) = v^{-1}$ satisfies the condition (A.1).

Let W be a Euclidean space of dimension m, Φ a representation of V in the space $Sym(W)$ of self adjoint endomorphism of W such that

$$\Phi(vv') = \frac{1}{2}\big(\Phi(v)\Phi(v') + \Phi(v')\Phi(v)\big), \quad v, v' \in V$$

and $Q : W \to V$ the quadratic mapping associated to Φ defined by

$$(Q(w)|v)_V = (\Phi(v)w|w)_W, \quad v \in V, w \in W. \tag{9}$$

Assume that Φ is *regular*, namely, there exists a $w \in W$ such that $\det Q(w) \neq 0$. Then the quadratic mapping Q is nondegenerate in the sense of (A.3) (i) (see Lemma 6 (1)), and we have $Q(W) = \overline{\Omega_1}$. We also assume that $\Phi(e) = \mathrm{id}_W$.

For an invertible $v \in V$, there exists a polynomial $q(v)$ of degree $r-1$ such that $v^{-1} = \frac{q(v)}{\det v}$ ([6, Prop. II.2.4]). Since Φ is a Jordan algebra representation, $\Phi(v)$ and $\Phi(v^{-1})$ commute. Hence

$$\mathrm{id}_W = \Phi(v \cdot v^{-1}) = \frac{1}{2}\big(\Phi(v)\Phi(v^{-1}) + \Phi(v^{-1})\Phi(v)\big) = \Phi(v)\Phi(v^{-1}).$$

This implies that Q is self-dual with respect to $\phi(v) = v^{-1}$.

Thus our Theorem 4 shows that the compositions of the fundamental relative invariants with Q satisfy a local functional equation. This recovers the results of Faraut-Koranyi and Clerc. Concrete examples are described in Clerc [5].

In [5], it is noted that, if the Jordan algebra V is of rank 2, then the generic norm det is a quadratic form of signature $(1, n-1)$ and the polynomials Q of degree 4 constructed as above are *not* relative invariants of prehomogeneous vector spaces except for some low-dimensional cases. However, it seems that no simple criterion on prehomogeneity has been known yet. For $n = \dim V$ with $n \leq 4$, all the polynomials obtained in this manner are relative invariants of some prehomogeneous vector spaces. For $n = 5$, there exists a unique simple V-module W_0 and its dimension is 8. If $W = W_0$, then $\tilde{P}$ is the square of a quadratic form of signature $(4, 4)$ and a relative invariant of the prehomogeneous vector space $(GL_1 \times SO(4,4), W_0)$. However, the direct sum $W = W_0 \oplus W_0$ gives a non-prehomogeneous example. Indeed, the group of linear transformations that leave the polynomial $\tilde{P}(w) = \det \circ Q(w)$ invariant is isomorphic to $Sp(1,1) \times Sp(2)$. By [17, §5, Proposition 21], the 16-dimensional representation of $GL(1) \times Sp(1,1) \times Sp(2)$ is not prehomogeneous. More generally, on the basis of calculation with a symbolic calculation engine done by Kogiso, we conjecture that

> if $W = \overbrace{W_0 \oplus \cdots \oplus W_0}^{k}$ $(k \geq 2)$, then the group of linear transformations that leave the polynomial $\tilde{P}(w) = \det \circ Q(w)$ invariant is isomorphic to $Sp(1,1) \times Sp(k)$ and hence $\tilde{P}$ is not a relative invariant of a prehomogeneous vector space.

This conjecture is proved for $k \leq 6$. A detailed analysis of this non-prehomogeneous example as well as its generalization will appear elsewhere as a joint work with T. Kogiso.

REMARK. In [5], Clerc proved local functional equations also for zeta functions with harmonic polynomials. This part is not covered by Theorem 4.

2.3. Some series of prehomogeneous vector spaces

EXAMPLE 1. Let V be the vector space of real square matrices of size r: $V = M_r(\mathbb{R})$. We identify V^* with $M_r(\mathbb{R})$ via the inner product $\langle v, v'\rangle = \mathrm{tr}({}^t v v')$. We denote by B_r the group of all nondegenerate real lower triangular matrices and put $G = B_r \times B_r$. The group G acts linearly on V by $\rho(b_1, b_2)v = b_1 v\, {}^t b_2$. The dual representation is given by $\rho^*(b_1, b_2)v = {}^t b_1^{-1} v b_2^{-1}$. For $i = 1, \dots, r$, denote by $d_i(v)$ (resp. $d_i^*(v)$) the determinant of the upper left (resp. lower right) i by i block of v. Then $d_1(v), \dots, d_r(v)$ (resp. $d_1^*(v), \dots, d_r^*(v)$) are the fundamental relative invariants of the prehomogeneous vector space (G, ρ, V) (resp. (G, ρ^*, V^*)). The mapping $\phi : \Omega \to \Omega^*$ defined by $\phi(v) = v^{-1}$ satisfies the condition (A.1).

Put $W = M_{k,r}(\mathbb{R}) \oplus M_{k,r}(\mathbb{R})$. We identify W^* with $M_{k,r}(\mathbb{R}) \oplus M_{k,r}(\mathbb{R})$ via the inner product $\langle (w_1, w_2), (w_1', w_2')\rangle = \mathrm{tr}({}^t w_1 w_1' + {}^t w_2 w_2')$. For a nondegenerate real square matrix Y of size k, we define quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$ by setting

$$Q(w) = Q(w_1, w_2) = {}^t w_1 Y w_2\,, \quad Q^*(w^*) = Q^*(w_1^*, w_2^*) = {}^t w_1^* Y^{-1} w_2^*\,.$$

Put $\tilde{G} = B_r \times B_r \times SL_k(\mathbb{R})$. The group $\tilde{G}$ acts linearly on W and W^* by

$$\tau(b_1, b_2, h)(w_1, w_2) = (h w_1\, {}^t b_1, Y^{-1}\, {}^t h^{-1} Y w_2\, {}^t b_2)\,,$$

$$\tau^*(b_1, b_2, h)(w_1^*, w_2^*) = ({}^t h^{-1} w_1^* b_1^{-1}, Y h Y^{-1} w_2^* b_2^{-1})\,.$$

It is clear that $Q_{\phi(v)}(w)$ and $Q_v^*(w^*)$ are dual to each other for $v = E_r$, the identity matrix. Hence, by Lemma 6, Q and Q^* are nondegenerate quadratic mappings and dual to each other with respect to ϕ. It is known that $(\tilde{G}, \tau, W)$ is a regular prehomogeneous vector space and $(\tilde{G}, \tau^*, W^*)$ is its dual, and $\tilde{P}_i(w) = d_i({}^t w_1 Y w_2)$ and $\tilde{P}_i^*(w^*) = d_i^*({}^t w_1^* Y^{-1} w_2^*)$ $(i = 1, \dots, r)$ are the fundamental relative invariants. Therefore the zeta functions attached to these polynomials satisfy a local functional equation by the general theory of prehomogenous vector spaces ([12]). The advantage of Theorem 4 in this case is that the gamma matrix $(\tilde{\Gamma}_{ij}(s))$ is given explicitly in terms of the gamma matrix $(\Gamma_{ij}(s))$ for the smaller prehomogeneous vector spaces (G, ρ, V) and (G, ρ^*, V^*). An explicit formula for the gamma matrix $(\Gamma_{ij}(s))$ was given in [14, p. 487, Theorem 3.4].

EXAMPLE 2. We put $V = Sym_r(\mathbb{R})$, the space of all real symmetric matrices of size r. We identify V^* with $Sym_r(\mathbb{R})$ via the inner product $\langle v, v'\rangle = \mathrm{tr}({}^t v v')$. The group $G = B_r$ acts linearly on V by $\rho(b)v = b v\, {}^t b$. The dual representation is given by $\rho^*(b)v = {}^t b^{-1} v b^{-1}$. Then $d_1(v), \dots, d_r(v)$ (resp. $d_1^*(v), \dots, d_r^*(v)$) are the fundamental relative invariants of the prehomogeneous vector space (G, ρ, V) (resp. (G, ρ^*, V^*)). The mapping $\phi : \Omega \to \Omega^*$ defined by $\phi(v) = v^{-1}$ satisfies the condition (A.1).

Put $W = M_{k,r}(\mathbb{R})$. We identify W^* with $M_{k,r}(\mathbb{R})$ via the inner product $\langle w, w' \rangle = \operatorname{tr}({}^t w w')$. For a nondegenerate real symmetric matrix Y of size k, we define quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$ by setting

$$Q(w) = Q(w) = {}^t w Y w, \quad Q^*(w^*) = Q^*(w^*, w^*) = {}^t w^* Y^{-1} w^* .$$

Put $\tilde{G} = B_r \times SO(Y)$. The group $\tilde{G}$ acts linearly on W and W^* by

$$\tau(b, h)w = hw^t b, \quad \tau^*(b, h)w^* = {}^t h^{-1} w^* b .$$

It is clear that $Q_{\phi(v)}(w)$ and $Q^*_v(w^*)$ are dual to each other for $v = E_r$, the identity matrix. Hence, by Lemma 6, Q and Q^* are nondegenerate quadratic mappings and dual to each other with respect to ϕ. It is known that $(\tilde{G}, \tau, W)$ is a regular prehomogeneous vector space and $(\tilde{G}, \tau^*, W^*)$ is its dual, and $\tilde{P}_i(w) = d_i({}^t w Y w)$ and $\tilde{P}^*_i(w^*) = d^*_i({}^t w^* Y^{-1} w^*)$ $(i = 1, \ldots, r)$ are the fundamental relative invariants. Theorem 4 gives an explicit formula for the gamma matrix $(\tilde{\Gamma}_{ij}(s))$ in terms of the gamma matrix $(\Gamma_{ij}(s))$ for (G, ρ, V) and (G, ρ^*, V^*). An explicit formula for the gamma matrix $(\Gamma_{ij}(s))$ was given in [14, p. 483, Theorem 3.2]. The gamma matrix $(\tilde{\Gamma}_{ij}(s))$ was calculated previously in [13, p. 190, Theorem 1] by using a different method.

If the symmetric matrix Y is positive definite, then the quadratic mapping is also obtained from the representation of the Jordan algebra $Sym_r(\mathbb{R})$ on $M_{k,r}(\mathbb{R})$ given by the right multiplication and is included in the works by Faraut, Koranyi and Clerc ([6], [5]).

A similar construction is possible also for

$$V = Alt_{2r}(\mathbb{R}), Herm_r(\mathbb{C}), Herm_r(\mathbb{H}), M_r(\mathbb{C}), M_r(\mathbb{H}) .$$

Not all of these cases are covered by the Faraut-Koranyi-Clerc theory; however they are different real forms of the ones appearing from their theory.

2.4. Quadratic mappings obtained from the spin representations

In this paragraph, using spin representations, we construct another family of quadratic mappings which gives non-prehomogeneous examples of local functional equations. As for properties of spin representations necessary to the following discussion, we refer to [4] (especially its Chap. III).

Let $n \geq 2$ be a positive integer and put

$$H_n = \begin{pmatrix} 0_n & E_n \\ E_n & 0_n \end{pmatrix},$$

where 0_n and E_n are the zero matrix and the identity matrix, respectively, of size n. We consider the quadratic form

$$P(x) = \frac{1}{2} {}^t x H_n x = \sum_{i=1}^{n} x_i x_{n+i} , \quad x = {}^t(x_1, \ldots, x_{2n}) \in V := \mathbb{R}^{2n} .$$

Denote by $\operatorname{Spin}(n, n)$ the spin group of P. Let (χ, V) be the vector representation of $\operatorname{Spin}(n, n)$. We identify V^* with $\mathbb{R}^{2n}$ via the inner product $\langle x, x' \rangle = {}^t x H_n x'$ $(x, x' \in \mathbb{R}^{2n})$.

Then $(GL_1(\mathbb{R}) \times \mathrm{Spin}(n,n), \rho, V)$ given by $\rho(t,h)x = t\chi(h)x$ is a regular prehomogeneous vector space with the fundamental relative invariant $P(x)$. The dual prehomogeneous vector space $(GL_1(\mathbb{R}) \times \mathrm{Spin}(n,n), \rho^*, V^*)$ is given by $\rho^*(t,h)x = t^{-1}\chi(h)x$ and the fundamental relative invariant is $P^*(x) = P(x)$. Moreover, the mapping

$$\phi : \Omega \longrightarrow \Omega^* = \Omega\,, \quad \phi(x) = \frac{1}{P(x)}x \tag{10}$$

satisfies the condition (A.1).

Let (θ_+, E_+) (resp. (θ_-, E_-)) be the even (resp. odd) half-spin representation of $\mathrm{Spin}(n,n)$. We put

$$E_+^* = \begin{cases} E_+ & (n \equiv 0 \pmod 2)\,, \\ E_- & (n \equiv 1 \pmod 2)\,, \end{cases} \quad E_-^* = \begin{cases} E_- & (n \equiv 0 \pmod 2)\,, \\ E_+ & (n \equiv 1 \pmod 2)\,. \end{cases}$$

Then there exists a nondegenerate pairing $\beta : E_\pm \times E_\pm^* \to \mathbb{R}$. Hence, if n is even, then $(\theta_\pm, E_\pm)$ can be viewed as the contragredient representation to itself. If n is odd, the contragredient to (θ_+, E_+) is (θ_-, E_-). Put $I = \{1, 2, \ldots, n\}$. For a subset J of I, the cardinality of J is denoted by $|J|$. Let $\{e_1, \ldots, e_{2n}\}$ be the standard basis of V. For a subset $J = \{j_1, \ldots, j_r\}$ $(j_1 < \cdots < j_r)$ of I, we define an element e_J and e_J^* in the Clifford algebra by

$$e_J = e_{j_1} \cdots e_{j_r}\,, \quad e_J^* = \mathrm{sgn}\begin{pmatrix} 1, 2, \ldots\ldots\ldots, n-1, n \\ j_r, \ldots, j_1, j_1', \ldots, j_{n-r}' \end{pmatrix} e_{j_1'} \cdots e_{j_{n-r}'}\,,$$

where $I - J = \{j_1', \ldots, j_{n-r}'\}$ $(j_1' < \cdots < j_{n-r}')$. Then, $\{e_J \mid J \subset I,\ |J| \equiv 0 \pmod 2\}$ is a basis of E_+ and $\{e_J^* \mid J \subset I,\ |J| \equiv 0 \pmod 2\}$ is the dual basis of E_+^* with respect to β. Similarly $\{e_J \mid J \subset I,\ |J| \equiv 1 \pmod 2\}$ is a basis of E_- and $\{e_J^* \mid J \subset I,\ |J| \equiv 1 \pmod 2\}$ is the dual basis of E_-^* with respect to β. We also put

$$\check{E}_+ = \begin{cases} E_- & (n \equiv 0 \pmod 2)\,, \\ E_+ & (n \equiv 1 \pmod 2)\,, \end{cases} \quad \check{E}_- = \begin{cases} E_+ & (n \equiv 0 \pmod 2)\,, \\ E_- & (n \equiv 1 \pmod 2)\,. \end{cases}$$

Then, it is known that there exists a nontrivial bilinear mapping $\gamma : E_\pm \times \check{E}_\pm \to V$ that is $\mathrm{Spin}(n,n)$-equivariant. The image of the mapping γ can be calculated explicitly with the following lemma.

LEMMA 7. *For two subsets J and K of $I := \{1, 2, \ldots, n\}$, we have*

$$\gamma(e_J, e_K) = \begin{cases} (-1)^{|J|(|J|+1)/2+\mu}\,\mathrm{sgn}\begin{pmatrix} I \\ J\setminus\{\mu\}, \mu, K\setminus\{\mu\} \end{pmatrix} \cdot e_\mu \\ \qquad\qquad \text{if } J \cup K = I \text{ and } J \cap K = \{\mu\}\,, \\ (-1)^{|K|(|K|+1)/2+(n+1)(n+2)/2+\mu}\,\mathrm{sgn}\begin{pmatrix} I\setminus\{\mu\} \\ K,\ J \end{pmatrix} \cdot e_{n+\mu} \\ \qquad\qquad \text{if } J \cup K = I \setminus \{\mu\} \text{ and } J \cap K = \emptyset\,, \\ 0 \qquad\qquad \text{otherwise}\,, \end{cases}$$

where, for a subset I' of I and a decomposition $I' = J \cup J' \cup J''$ of I' into a disjoint union of subsets, we denote by $\begin{pmatrix} I' \\ J, J', J'' \end{pmatrix}$ *the permutation of I' obtained by placing the elements of I', J, J', J'' in the ascending order.*

For $k \geq 1$, put

$$W = (E_+ \oplus \check{E}_+)^{\oplus k}, \quad W^* = (E_+^* \oplus \check{E}_+^*)^{\oplus k}.$$

Then W^* is the dual of W^*. Now we can define a Spin(n, n)-equivariant quadratic mapping $Q : W \to V$ by

$$Q((w_1^{(1)}, w_2^{(1)}), \ldots, (w_1^{(k)}, w_2^{(k)})) = \sum_{i=1}^{k} \gamma(w_1^{(i)}, w_2^{(i)}).$$

We define $Q^* : W^* \to V$ by

$$Q^*((w_1^{*(1)}, w_2^{*(1)}), \ldots, (w_1^{*(k)}, w_2^{*(k)})) = (-1)^{n(n-1)/2} \cdot 2 \sum_{i=1}^{k} \gamma(w_1^{*(i)}, w_2^{*(i)}).$$

PROPOSITION 8. *The mappings $Q : W \to V$ and $Q^* : W^* \to V$ are dual to each other with respect to ϕ given in* (10). *If $n \geq 4$, or $n = 2, 3$, $k \geq 2$, then they are nondegenerate.*

Proof. Using Lemma 7, one can check the duality at $v = e_1 + e_{n+1}$ by direct calculation of the matrices of Q_v and Q_v^*. By Lemma 6 (2), this implies the duality at every point in $\Omega = \{w \in W \mid \tilde{P}(w) \neq 0\}$. When $n \geq 4$, we have

$$\gamma(1 + e_1e_2e_3e_4, e_2e_3 \cdots e_n + e_1e_5 \cdots e_n) = e_1 + (-1)^n e_{n+1}.$$

This implies that $\tilde{P}(w) = P(Q(w))$ does not vanish identically. Thus, by Lemma 6 (1), Q is nondegenerate for $n \geq 4$. We omit the similar proof for Q^*. The case $n = 2, 3$ will be discussed separately below. □

Let us examine some low dimensional cases. In case $n \geq 4$, we shall obtain new non-prehomogeneous examples of polynomials satisfying local functional equations.

Case $n = 2$ **:** In this case we have an isomorphism Spin$(2, 2) \cong SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ and the half-spin representations are given by $\theta_+(h_1, h_2)v = h_1 v$ and $\theta_-(h_1, h_2)v = h_2 v$. If $k = 1$, the explicit form of the mapping $Q : E_+ \oplus E_- \to \mathbb{R}^4$ is as follows:

$$Q : \mathbb{R}^2 \oplus \mathbb{R}^2 \to M_2(\mathbb{R}), \quad q(w_1, w_2) = w_1\, {}^t w_2.$$

Hence, we identify V with $M_2(\mathbb{R})$ and $P(v)$ with $\det v$. Since $\tilde{P}(w_1, w_2) = \det(Q(w_1, w_2)) = 0$, the mapping Q is degenerate. Now assume that $k \geq 2$. Then the mapping Q is given by

$$Q : M_{2,k}(\mathbb{R}) \oplus M_{2,k}(\mathbb{R}) \to M_2(\mathbb{R}), \quad Q(w_1, w_2) = w_1\, {}^t w_2$$

and the polynomial $\tilde{P}(w_1, w_2) = \det(w_1\, {}^t w_2)$ vanishes for $k = 1$ and, if $k \geq 2$, it gives the fundamental relative invariant of a prehomogeneous vector space ($GL_2 \times GL_2 \times$

$GL_k, M_{2,k}^{\oplus 2})$. Hence Q is nondegenerate if $k \geq 2$. This is a special case of Section 1.4.3, Example 1.

Case $n = 3$ **:** In this case we have an isomorphism $\mathrm{Spin}_6(\mathbb{R}) \cong SL_4(\mathbb{R})$ and the half-spin representations are given by $(\theta_+, E_+) = (\Lambda_1, \mathbb{R}^4)$ and $(\theta_-, E_-) = (\Lambda_3, \mathbb{R}^4)$. The vector representation is given by $(\chi, V) = (\Lambda_2, Alt_4(\mathbb{R}))$. Here we denote by Λ_i $(i = 1, 2, 3)$ the fundamental representations of SL_4 and by $Alt_4(\mathbb{R})$ the vector space of real alternating matrices of size 4. We identify V with $Alt_4(\mathbb{R})$ and $P(v)$ with the Pfaffian $Pf(v)$ of $v \in Alt_4(\mathbb{R})$. If we identify $E_+ \oplus E_-$ with $M_{4,2}(\mathbb{R})$ and $(E_+ \oplus E_-)^{\oplus k}$ with $M_{4,2k}(\mathbb{R})$, then the mapping $Q : (E_+ \oplus E_-)^{\oplus k} \to Alt_4(\mathbb{R})$ is given by

$$Q : M_{4,2k}(\mathbb{R}) \to Alt_4(\mathbb{R}), \quad w \mapsto wJ_k\,{}^t w, \quad J_k = \overbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \perp \cdots \perp \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}^{k}$$

and the polynomial $\tilde{P}(w) = Pf(w_1 J_k\,{}^t w_2)$ vanishes for $k = 1$ and, if $k \geq 2$, it gives the fundamental relative invariant of a prehomogeneous vector space $(GL_4 \times Sp_{2k}, M_{4,2k})$. Hence Q is nondegenerate if $k \geq 2$. Thus we have completed the proof of Proposition 8.

Case $n = 4$ **:** In this case, by the principle of triality, the group of outer automorphisms of Spin(4, 4) permutes the 3 representations θ_+, θ_-, χ and they are on the same footing. Therefore we have 2 quadratic forms q_+ and q_- on E_+ and E_- in 8 variables, which are Spin(4, 4)-invariant. For $k = 1$, the polynomial $\tilde{P}(w) = P(Q(w))$ $(w \in W = E_+ \oplus E_-)$ is a Spin(4, 4)-invariant polynomial of degree 4. By direct calculation, we can easily identify $\tilde{P}$ with the product $q_+ q_-$. Therefore $\tilde{P}$ is a (not fundamental) relative invariant of the prehomogeneous vector space $(GL_1 \times \mathrm{Spin}(4,4), \Lambda_1 \otimes \theta_+, E_+) \oplus (GL_1 \times \mathrm{Spin}(4,4), \Lambda_1 \otimes \theta_-, E_-)$, where Λ_1 denotes the scalar multiplication. However, if $k \geq 2$, it seems that $\tilde{P}$ is *not* a relative invariant of any prehomogeneous vector space. Indeed, let us consider the case $k = 2$. Let G^1 be the group of linear transformations on W that leave $\tilde{P}$ invariant. Then the Lie algebra $\mathfrak{g}^1$ of G^1 is isomorphic to $\mathfrak{so}(4,4) \oplus \mathfrak{gl}(2,\mathbb{R})$ and the representation of $\mathfrak{g}^1$ on the 32-dimensional vector space $W = (E_+ \oplus E_+) \oplus (E_- \oplus E_-)$ is given by $\theta_+ \otimes \Lambda_1 + \theta_- \otimes \Lambda_1^*$, where Λ_1 denotes the standard 2-dimensional representation of $GL_2(\mathbb{R})$ and Λ_1^* its dual. This representation (together with scalar multiplications) can not be prehomogeneous by [10, Theorem 2.16].

Case $n = 5$ **:** In this case, if $k = 1$, then the polynomial $\tilde{P}(w) = P(Q(w))$ is the fundamental relative invariant of the prehomogeneous vector space $(\mathrm{Spin}(5,5) \times GL_2, \theta_+ \otimes \Lambda_1)$, where Λ_1 denotes the standard 2-dimensional representation of $GL_2(\mathbb{R})$. This construction of the fundamental relative invariant coincides with the one given by H. Kawahara in his Master Thesis (University of Tokyo, 1974). If $k \geq 2$, it seems that $\tilde{P}$ is *not* a relative invariant of any prehomogeneous vector space.

Case $n \geq 6$ **:** In this case, it seems that $\tilde{P}$ is *not* a relative invariant of any prehomogeneous vector space for arbitrary $k \geq 1$. As an example let us consider the case where $n = 6$ and $k = 1$. As in the case $n = 4, k = 2$, let G^1 be the the group of linear transformations on W that leave $\tilde{P}$ invariant. Then the Lie algebra $\mathfrak{g}^1$ of G^1 is isomorphic to $\mathfrak{so}(6,6) \oplus \mathfrak{gl}(1,\mathbb{R})$ and the representation of $\mathfrak{g}^1$ on the 64-dimensional vector space $W = E_+ \oplus E_-$ is given by $\theta_+ \otimes \Lambda_1 + \theta_- \otimes \Lambda_1^*$. Here Λ_1 is the scalar multiplication and Λ_1^* is its dual. This representation (together with scalar multiplications) can not be prehomogeneous by [8, Proposition 2.32].

3. Proof of the main theorem

In this section we give a proof of Theorem 4.

3.1. Weak functional equations

We consider the following weak version of the condition (A.2):

(A.2b) There exist an $A \in GL_r(\mathbb{Z})$ and a $\lambda \in \mathbb{C}^r$ such that a functional equation of the form

$$\zeta_i^*((s+\lambda)A, \hat{\Phi}) = \sum_{j=1}^{\nu} \Gamma_{ij}(s)\zeta_j(s,\Phi) \quad (i = 1,\ldots,\nu) \tag{11}$$

holds for every $\Phi \in \mathcal{S}(V)$ with the property that $\Phi \in C_0^\infty(\Omega)$ or $\hat{\Phi} \in C_0^\infty(\Omega^*)$. Here, as in (A.2), $\Gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}^r$ not depending on Φ with $\det(\Gamma_{ij}(s)) \not\equiv 0$.

The aim of this subsection is to show that this weaker condition (A.2b) is actually equivalent to the apparently stronger condition (A.2). This equivalence of the two conditions played a crucial role in the proof of local functional equations in the theory of prehomogeneous vector spaces (see [15], [19], [12]). In the prehomogeneous case, Ω_i and Ω_j^* are orbits of a Lie group and the weak form (A.2b) of the functional equation is an immediate consequence of the uniqueness of relatively invariant distributions on a homogeneous space. The essential part of the proof is the derivation of (A.2) from (A.2b), which is irrelevant to any group action and can be generalized to the present situation. The argument here is the same as that in [19] and [12, §5]. We include it only for the same of completeness.

First we prove Lemma 1, the existence of the b-functions, under the assumption (A.2b).

LEMMA 9. *Assume that the condition* (A.2b) *is satisfied. For* $\mathbf{m} \in \mathbb{Z}^n$ *with* $m_1,\ldots,m_r \geq 0$, *denote by* $P^{*\mathbf{m}}(\partial_v)$ *the linear partial differential operator with constant coefficients satisfying*

$$P^{*\mathbf{m}}(\partial_v)\exp(\langle v, v^*\rangle) = P^{*\mathbf{m}}(v^*)\exp(\langle v, v^*\rangle)\,.$$

Then, there exists a polynomial $b_{\mathbf{m}}(s)$ *of* $s_1,\ldots,s_r$ *such that*

$$P^{*\mathbf{m}}(\partial_v)P(v)^s = b_{\mathbf{m}}(s)P(v)^{s+\mathbf{m}'}\,, \quad \mathbf{m}' = \mathbf{m}A^{-1}\,. \tag{12}$$

Moreover, if $\Gamma_{ij}(s)$ does not vanish identically, then the polynomial $b_{\mathbf{m}}(s)$ is given by

$$b_{\mathbf{m}}(s) = (-2\pi\sqrt{-1})^{d^*(\mathbf{m})}\epsilon_j(\mathbf{m}')\epsilon_i^*(\mathbf{m}) \cdot \frac{\Gamma_{ij}(s+\mathbf{m}')}{\Gamma_{ij}(s)}\,, \tag{13}$$

where $d^*(\mathbf{m}) = \deg P^{*\mathbf{m}}(v^*)$, and $\epsilon_i(\mathbf{m}')$ (resp. $\epsilon^*(\mathbf{m})$) is the sign of $P^{\mathbf{m}'}(v)$ (resp. $P^{*\mathbf{m}}(v^*)$) on Ω_i (resp. Ω_j^*).

Proof. Let Φ_0 be a function in $\mathcal{S}(V)$ whose support is contained in Ω_i. Put $\Phi(v) = P^{*\mathbf{m}}(\partial_v)\Phi_0(v)$. The support of Φ is also contained in Ω_i. Note that $\zeta_k(s,\Phi) = 0$ unless $i = k$. Denote by $\hat{\Phi}_0$ the Fourier transform of Φ_0. The Fourier transform $\hat{\Phi}$ of Φ is given by

$$\hat{\Phi}(v^*) = (2\pi\sqrt{-1})^{d^*(\mathbf{m})}P^{*\mathbf{m}}(v^*)\hat{\Phi}_0(v^*)\,.$$

By (A.2b), we have

$$\zeta_i^*((s+\lambda)A, \hat{\Phi}) = \Gamma_{ij}(s)\zeta_j(s,\Phi)\,.$$

Since

$$\zeta_i^*((s+\lambda)A, \hat{\Phi}) = (2\pi\sqrt{-1})^{d^*(\mathbf{m})}\epsilon_i^*(\mathbf{m})\zeta_i^*((s+\lambda+\mathbf{m}')A, \hat{\Phi}_0)\,,$$

we have

$$\zeta_i^*((s+\lambda)A, \hat{\Phi}) = (2\pi\sqrt{-1})^{d^*(\mathbf{m})}\epsilon_i^*(\mathbf{m})\Gamma_{ij}(s+\mathbf{m}')\zeta_j(s+\mathbf{m}', \Phi_0)\,.$$

On the other hand, we have

$$\zeta_i(s,\Phi) = (-1)^{d^*(\mathbf{m})}\int_V \left(P^{*\mathbf{m}}(\partial_v)|P(v)|_i^s\right)\Phi_0(v)\,dv\,.$$

Hence the integral

$$\int_V \left\{P^{*\mathbf{m}}(\partial_v)(|P(v)|_i^s) - (-2\pi\sqrt{-1})^{d^*(\mathbf{m})}\epsilon_i^*(\mathbf{m}) \cdot \frac{\Gamma_{ij}(s+\mathbf{m}')}{\Gamma_{ij}(s)}|P(v)|_i^{s+\mathbf{m}'}\right\}\Phi_0(v)\,dv$$

vanishes for every $\Phi_0 \in C_0^\infty(\Omega_i)$. Hence we have the identity (13). Since $P^{*\mathbf{m}}(\partial_v)$ $P^s(v)/P^{s+\mathbf{m}'}(v)$ evaluated at a point $v \in \Omega_i$ is in $\mathbb{C}[s]$, the function $b_{\mathbf{m}}(s)$ is a polynomial of s. □

PROPOSITION 10. *Assume the condition* (A.2b). *Then the functional equation* (11) *holds for every $\Phi \in \mathcal{S}(V)$.*

Proof. For $\Phi^* \in \mathcal{S}(V^*)$, we define the inverse Fourier transform $\check{\Phi}^*$ by

$$\check{\Phi}^*(v) = \int_{V^*}\Phi^*(v^*)\exp(2\pi\sqrt{-1}\langle v, v^*\rangle)\,dv^*\,.$$

The condition (A.2b) means that the support of the tempered distribution

$$T_s : \mathcal{S}(V^*) \ni \Phi^* \longmapsto \zeta_i^*((s+\lambda)A, \Phi^*) - \sum_{j=1}^{\nu}\Gamma_{ij}(s)\zeta_j(s,\check{\Phi}^*) \in \mathbb{C}$$

is containd in $V^* - \Omega^* = \{v^* \in V^* \mid (P_1^* \cdots P_r^*)(v^*) = 0\}$. Take an $s_0 \in \mathbb{C}^n$ such that T_s is holomorphic at $s = s_0$. Since the order of the distribution T_s is locally bounded (see [12, Lemma 5.2 (iii)]), one can find an $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ with $m_1, \ldots, m_r \geq 0$ such that $P^{*\mathbf{m}} T_s$ is identically 0 for all s in a sufficiently small neighborhood of s_0 (see [19, Lemma 1.3]). This implies that, for every $\Phi^* \in \mathcal{S}(V^*)$, we have

$$\zeta_i^*((s+\lambda)A, P^{*\mathbf{m}}\Phi^*) = \sum_{j=1}^{\nu} \Gamma_{ij}(s)\zeta_j(s, (P^{*\mathbf{m}}\Phi^*)^\vee). \tag{14}$$

We have

$$\zeta_i^*((s+\lambda)A, P^{*\mathbf{m}}\Phi^*) = \epsilon_i^*(\mathbf{m})\zeta_i^*((s+\mathbf{m}'+\lambda)A, \Phi^*)$$

and, by (12),

$$\begin{aligned}\zeta_j(s, (P^{*\mathbf{m}}\Phi^*)^\vee) &= (-2\pi\sqrt{-1})^{d^*(\mathbf{m})}\zeta_j(s, P^{*\mathbf{m}}(\partial_v)\check{\Phi}^*)\\ &= (2\pi\sqrt{-1})^{-d^*(\mathbf{m})} b_{\mathbf{m}}(s)\epsilon_j(\mathbf{m}')\zeta_j(s+\mathbf{m}', \check{\Phi}^*).\end{aligned}$$

Hence the identity (14) takes the form

$$\zeta_i^*((s+\mathbf{m}'+\lambda)A, \Phi^*) = \sum_{j=1}^{\nu} \Gamma_{ij}(s)(2\pi\sqrt{-1})^{-d^*(\mathbf{m})} b_{\mathbf{m}}(s)\epsilon_j(\mathbf{m}')\epsilon_i^*(\mathbf{m})\zeta_j(s+\mathbf{m}', \check{\Phi}^*).$$

By (13), we have

$$\zeta_i^*((s+\mathbf{m}'+\lambda)A, \Phi^*) = \sum_{j=1}^{\nu} \Gamma_{ij}(s+\mathbf{m}')\zeta_j(s+\mathbf{m}', \check{\Phi}^*)$$

for all $\Phi^* \in \mathcal{S}(V^*)$. □

3.2. Proof of Theorem 4

By Proposition 10, it is sufficient to prove the theorem for Ψ satisfying $\Psi \in C_0^\infty(\tilde{\Omega})$ or $\check{\Psi} \in C_0^\infty(\tilde{\Omega}^*)$. Since the proof is similar, we give a proof only in the case $\check{\Psi} \in C_0^\infty(\tilde{\Omega}^*)$. By the nondegeneracy of the quadratic mappings Q and Q^*, we have

$$\tilde{\zeta}_i^*((s+2\lambda+\kappa/2+\mu)A, \check{\Psi}) = \int_{\Omega_i^*} |P^*(v^*)|^{(s+2\lambda+\kappa/2+\mu)A} M(\check{\Psi}, v^*)\, dv^*,$$

$$M(\check{\Psi}, v^*) = \int_{W^*} \check{\Psi}(w^*)\delta(Q^*(w^*)-v^*)\, dw^*.$$

Since the support of $\check{\Psi}$ is a compact subset of $\tilde{\Omega}^* = (Q^*)^{-1}(\Omega^*)$, the function $M(\check{\Psi}, v^*)$ is in $C_0^\infty(\Omega^*)$. Hence the inverse Fourier transform of $M(\check{\Psi}, v^*)$ is in $\mathcal{S}(V)$ and is given by

$$\begin{aligned}\check{M}(\check{\Psi}, v) :&= \int_{V^*} M(\check{\Psi}, v^*)\exp(2\pi\sqrt{-1}\langle v, v^*\rangle)\, dv\\ &= \int_{W^*} \check{\Psi}(w^*)\exp(2\pi\sqrt{-1}\langle v, Q^*(w^*)\rangle)\, dw^*.\end{aligned}$$

Hence, applying the functional equation (A.2), we obtain

$$\begin{aligned}&\tilde{\zeta}_i^*((s+2\lambda+\kappa/2+\mu)A,\check{\Psi})\\&=\sum_{k=1}^{\nu}\Gamma_{ik}(s+\lambda+\kappa/2+\mu)\int_V|P(v)|_k^{s+\lambda+\kappa/2+\mu}\check{M}(\check{\Psi},v)\,dv\,.\end{aligned}$$

From now on we assume that the real parts of s are sufficiently large. Then the integral on the right hand side of the identity above is absolutely convergent. By (5), we have for $v\in\Omega_k$

$$\begin{aligned}\check{M}(\check{\Psi},v)&=\int_{W^*}\check{\Psi}(w^*)\exp(2\pi\sqrt{-1}Q_v^*(w^*))\,dw^*\\&=2^{-m/2}\gamma_k|\alpha|^{1/2}|P(v)|_k^{-\kappa/2}\int_W\Psi(w)\exp\left(-\frac{\pi\sqrt{-1}}{2}Q_{\phi(v)}(w)\right)dw\,.\end{aligned}$$

Hence, putting

$$\check{M}(\Psi,v^*)=\int_W\Psi(w)\exp\left(-\frac{\pi\sqrt{-1}}{2}Q_{v^*}(w)\right)d\,w\,,$$

we have

$$\check{M}(\check{\Psi},v)=2^{-m/2}\gamma_k|\alpha|^{1/2}|P(v)|_k^{-\kappa/2}\check{M}(\Psi,\phi(v))\,,\qquad v\in\Omega_k\,.$$

Therefore, by Lemma 2, putting $v^*=\phi(v)$, we obtain

$$\begin{aligned}&\tilde{\zeta}_i^*((s+2\lambda+\kappa/2+\mu)A,\check{\Psi})\\&=2^{-m/2}|\alpha|^{1/2}\sum_{k=1}^{\nu}\gamma_k\Gamma_{ik}(s+\lambda+\kappa/2+\mu)\int_{\Omega_k^*}|P^*(v^*)|^{(s+\lambda)A}\check{M}(\Psi,v^*)\,dv^*\,.\end{aligned}$$

Note that the integral on the right hand side of the identity is absolutely convergent when the real parts of s are sufficiently large and $\check{M}(\Psi,v^*)$ is a bounded C^∞-function on V^*. Hence

$$\begin{aligned}&\int_{\Omega_k^*}|P^*(v^*)|^{(s+\lambda)A}\check{M}(\Psi,v^*)\,dv^*\\&\qquad=\lim_{\epsilon\downarrow 0}\int_{\Omega_k^*}|P^*(v^*)|^{(s+\lambda)A}\exp(-\epsilon\pi||v^*||^2)\check{M}(\Psi,v^*)\,dv^*\,,\end{aligned}$$

where $||v^*||$ is the length of a vector v^*. The function $\exp(-\epsilon\pi||v^*||^2)\check{M}(\Psi,v^*)$ is a Schwartz function on V^* and its inverse Fourier transform is given by

$$\begin{aligned}&\int_{V^*}\exp(-\epsilon\pi||v^*||^2+2\pi\sqrt{-1}\langle v,v^*\rangle)\check{M}(\Psi,v^*)\,dv^*\\&=\int_{V^*}\exp(-\epsilon\pi||v^*||^2+2\pi\sqrt{-1}\langle v,v^*\rangle)dv^*\int_W\Psi(w)\exp\left(-\frac{\pi\sqrt{-1}}{2}Q_{v^*}(w)\right)dw\\&=\int_W\Psi(w)\,dw\int_{V^*}\exp\left(-\epsilon\pi||v^*||^2+2\pi\sqrt{-1}\left\langle v^*,v-\frac{1}{4}Q(w)\right\rangle\right)dv^*\end{aligned}$$

$$= \epsilon^{-n/2}\int_W \Psi(w)\exp\left(-(\pi/\epsilon)\left\|v-\frac{1}{4}Q(w)\right\|^2\right)dw.$$

Hence, applying the functional equation in (A.2) once again, we have

$$\int_{\Omega_k^*}|P^*(v^*)|^{(s+\lambda)A}\check{M}(\Psi, v^*)\,dv^*$$
$$= \lim_{\epsilon\downarrow 0}\epsilon^{-n/2}\sum_{j=1}^{\nu}\Gamma_{kj}(s)\int_{\Omega_j}|P(v)|^s\,dv\int_W \Psi(w)\exp\left(-(\pi/\epsilon)\left\|v-\frac{1}{4}Q(w)\right\|^2\right)dw\,.$$

By the Lebesgue convergence theorem, we have

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\epsilon^{-n/2}\int_{\Omega_j}|P(v)|^s\,dv\int_W \Psi(w)\exp\left(-(\pi/\epsilon)\left\|v-\frac{1}{4}Q(w)\right\|^2\right)dw\\
&= \lim_{\epsilon\downarrow 0}\epsilon^{-n/2}\int_W \Psi(w)\,dw\int_V\left|P\left(v+\frac{1}{4}Q(w)\right)\right|_j^s\exp(-(\pi/\epsilon)||v||^2)\,dv\\
&= \lim_{\epsilon\downarrow 0}\int_W\int_V \Psi(w)\left|P\left(\epsilon^{1/2}v+\frac{1}{4}Q(w)\right)\right|_j^s\exp(-\pi||v||^2)\,dv\,dw\\
&= 2^{-2d(s)}\int_V\exp(-\pi||v||^2)\,dv\int_W|P(Q(w))|_j^s\Psi(w)\,dw\\
&= 2^{-2d(s)}\tilde{\zeta}_j(s,\Psi)\,,
\end{aligned}$$

where $d(s)$ is the same as in Theorem 4. Thus we obtain

$$\begin{aligned}
&\tilde{\zeta}_i^*((s+2\lambda+\kappa/2+\mu)A,\check{\Psi})\\
&= 2^{-2d(s)-m/2}|\alpha|^{1/2}\sum_{j=1}^{\nu}\left(\sum_{k=1}^{\nu}\gamma_k\Gamma_{ik}(s+\lambda+\kappa/2+\mu)\Gamma_{kj}(s)\right)\tilde{\zeta}_j(s,\Psi)\,.
\end{aligned}$$

This proves the theorem.

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